ON $\overline{\partial}$ HOMOTOPY FORMULAE FOR PRODUCT DOMAINS: NIJENHUIS-WOOLF'S FORMULAE AND OPTIMAL SOBOLEV ESTIMATES

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Abstract. We construct homotopy formulae $f = \overline{\partial}\mathcal{H}_q f + \mathcal{H}_{q+1}\overline{\partial} f$ for $(0,q)$ forms on the product domain $\Omega_1 \times \cdots \times \Omega_m$, where each Ω_j is either a bounded Lipschitz domain in $\mathbb{C}^1$, a bounded strongly pseudoconvex domain with C^2 boundary, or a smooth convex domain of finite type. Such homotopy operators $\mathcal{H}_q$ yield solutions to the $\overline{\partial}$ equation with optimal Sobolev regularity $W^{k,p} \to W^{k,p}$ simultaneously for all $k \in \mathbb{Z}$ and $1 < p < \infty$.

1. Introduction

The goal of this paper is to prove the following:

Theorem 1. *Let $\Omega_j \subset \mathbb{C}^{n_j}$ be a bounded Lipschitz domain for each $j = 1, \dots, m$, with $m \geq 1$, such that one of the following holds.*

- *$\Omega_j \subset \mathbb{C}$ is a planar domain (i.e. $n_j = 1$).*
- *Ω_j is strongly pseudoconvex with C^2 boundary or strongly $\mathbb{C}$-linearly convex with $C^{1,1}$ boundary.*
- *Ω_j is a smooth convex domain of finite type.*

Let $\Omega := \Omega_1 \times \cdots \times \Omega_m$ and $n := \sum_{j=1}^m n_j$. Then there exist linear operators $\mathcal{P} = \mathcal{P}^\Omega : \mathscr{S}'(\Omega) \to \mathscr{S}'(\Omega)$ and $\mathcal{H}_q = \mathcal{H}_q^\Omega : \mathscr{S}'(\Omega; \wedge^{0,q}) \to \mathscr{S}'(\Omega; \wedge^{0,q-1})$ for $1 \leq q \leq n$, such that

(i) $f = \mathcal{P} f + \mathcal{H}_1 \overline{\partial} f$ for all $f \in \mathscr{S}'(\Omega)$; and $f = \overline{\partial}\mathcal{H}_q f + \mathcal{H}_{q+1}\overline{\partial} f$ for all $f \in \mathscr{S}'(\Omega; \wedge^{0,q})$.

(ii) We have Sobolev estimates $\mathcal{P} : W^{k,p}(\Omega) \to W^{k,p}(\Omega)$ and $\mathcal{H}_q : W^{k,p}(\Omega; \wedge^{0,q}) \to W^{k,p}(\Omega; \wedge^{0,q-1})$ for all $k \in \mathbb{Z}$ and $1 < p < \infty$.

Here $\mathscr{S}'(\Omega)$ is the space of distributions on Ω which admit extension to distributions on $\mathbb{C}^n$, $W^{k,p}(\Omega)$ is the Sobolev space on Ω with $k \in \mathbb{Z}$ and $1 < p < \infty$, and $\mathscr{S}'(\Omega; \wedge^{0,q})$ (resp. $W^{k,p}(\Omega; \wedge^{0,q})$) is the space of degree $(0,q)$ forms with coefficients in $\mathscr{S}'(\Omega)$ (resp. $W^{k,p}(\Omega)$). See Notation 4, Definition 5 and Convention 13 for the precise definitions. We note that $(\mathcal{H}_q)_{q=1}^n$ do not possess optimal Hölder estimates. See Remark 28.

As an immediate consequence of Theorem 1 we obtain a solution operator to the $\overline{\partial}$ equation on product domains for any $(0,q)$ forms with $q \geq 1$, together with the associated Sobolev estimate. Note that in view of a Kerzman-type example (see, e.g. [Zha24, Example 1]) the Sobolev regularity is sharp.

Corollary 2. *Let $\Omega \subset \mathbb{C}^n$ be given as in Theorem 1. Let $1 \leq q \leq n$. For every $(0,q)$-form f on Ω whose coefficients are extendable distributions such that $\overline{\partial} f = 0$ in the sense of distributions, there is a $(0,q-1)$ form u on Ω whose coefficients are also extendable distributions, such that $\overline{\partial} u = f$.*

Moreover, for every $k \in \mathbb{Z}$ and $1 < p < \infty$ there is a constant $C > 0$, such that if the coefficients of f are in $W^{k,p}$, then we can choose u whose coefficients are in $W^{k,p}$, with the estimate

$$\|u\|_{W^{k,p}(\Omega; \wedge^{0,q-1})} \leq C \|f\|_{W^{k,p}(\Omega; \wedge^{0,q})}.$$

The $\overline{\partial}$ homotopy formulae play an essential role in studying the $\overline{\partial}$ problem and have been extensively developed of pseudoconvex domains with certain finite type conditions, using the $\overline{\partial}$-Neumann approach (e.g. [GS77, FK88, Cha89, CNS92]) and the integral representation approach (e.g. [LR80, Ran90,

2020 *Mathematics Subject Classification.* 32A26 (primary) 32W05 and 46E35 (secondary).

Key words and phrases. Cauchy-Riemann equations, integral representation, product of pseudoconvex domains, negative Sobolev spaces.

DFF99, Gon19]). We refer the readers to [Ran90, CS01, LM02] for more details. Product domains, owing to their particular structure, fail to have finite type, and merely admit Lipschitz boundary regularity.

The study of the $\overline{\partial}$ problem on product domains was initiated by the work of Henkin [Hen71], who established L^∞ estimates for the $\overline{\partial}$ equation on the bidisk with continuous coefficients using an integral representation by the Cauchy kernel. Landucci [Lan75] later proved an analogous result for the canonical solutions. Since the work of Chen-McNeal [CM20b] on weighted L^p estimates for product domains in $\mathbb{C}^2$, there has been much progress towards the optimal L^p estimates for $(0,1)$ forms on Cartesian products of general planar domains. The important case $p=\infty$ for $(0,1)$ forms (on planar domains) without weights is also called the Kerzman's supnorm estimate problem, posted by Kerzman [Ker71] in 1971. The optimal L^∞ estimate was first given by Fassina-Pan [FP24] for forms with $C^{n-1,\alpha}$ coefficients. Later Dong-Pan-Zhang [DPZ20] extended this result to the canonical solutions for continuous data. Finally, Kerzman's problem was completely solved recently by Yuan [Yua22] on products of C^2 planar domains based on [DPZ20], and Li [Li24] on products of $C^{1,\alpha}$ planar domains independently. Subsequently Li-Long-Luo [LLL24] further relaxed the boundary regularity of each Ω_j to be Lipschitz. The Sobolev regularity of $\overline{\partial}$ was first investigated by Chakrabarti-Shaw [CS11] for the canonical solutions with respect to $(0,1)$ forms on products of certain smooth pseudoconvex domains. In particular, the optimal $W^{k,p}, k\geq 1$ regularity was obtained for products of smooth planar domains in $\mathbb{C}^2$ by Jin-Yuan [JY20] and in $\mathbb{C}^n$ by Zhang [Zha24]. See also [Jak86, DLT23, CM20a, PZ25] and the references therein.

In comparison to those results, our theorem allows each factor $\Omega_j\subset\mathbb{C}^{n_j}$ to be non-planar (i.e. we allow $n_j>1$). Such product domains were previously studied in [Jak86, CS11, CM20a] for $(0,1)$ forms in certain special Sobolev spaces that are strictly smaller than the standard ones and involve a loss of derivatives. In the special case of planar product domains, we obtain Sobolev estimates assuming that each factor Ω_j to be merely Lipschitz, which extends the $L^p\to L^p$ estimate from [LLL24]. Meanwhile, we show the existence of $\overline{\partial}$-solutions on space of distributions, a result that is novel even for polydisks. Previously the similar solvability on distributions (with large orders) were only known for strongly pseudoconvex domains by Shi-Yao [SY24b] and Yao [Yao24b] and for convex domains of finite type by Yao [Yao24a]. Moreover, we derive the optimal estimates for general $(0,q)$ forms with all $q\geq 1$. To the authors' best knowledge, for the case $q\geq 2$, the only previously established result for optimal $\overline{\partial}$ estimate on product domains is the $L^2\to L^2$ estimate, which follows directly from Hörmander's classical L^2-theory. In particular, the L^p-boundedness for $p\neq 2$ remains open for the canonical solutions on $(0,q)$ forms even on polydisk.

Our construction of the homotopy operators is inspired from Nijenhuis-Woolf's formulae in [NW63, (2.2.2)] for products of planar domains, see Theorem 18 and Remark 23. For estimates we use the so-called Fubini decomposition of Sobolev spaces, see Proposition 10.

In fact, the proof yields homotopy formulae and the corresponding operator estimates on a considerably larger class of product domains, provided that each factor domain admits its homotopy formulae and regularity estimates. To be precise, the following is the conditional result:

Theorem 3. *Let $\Omega_j\subset\mathbb{C}^{n_j}$ be a bounded Lipschitz domain for each $j=1,\dots,m$, with $m\geq 1$. Suppose there exist linear homotopy operators $H_q^{\Omega_j}: C^\infty(\overline{\Omega_j};\wedge^{0,q})\to\mathscr{D}'(\Omega_j;\wedge^{0,q-1})$ for $1\leq q\leq n_j$, such that the following homotopy formulae hold (we set $H_{n_j+1}^{\Omega_j}=0$):*

$$f=\overline{\partial}H_q^{\Omega_j}f+H_{q+1}^{\Omega_j}\overline{\partial}f\quad\text{for all}\quad f\in C^\infty(\overline{\Omega_j};\wedge^{0,q}),\qquad 1\leq q\leq n_j. \tag{1}$$

Let $\Omega:=\Omega_1\times\cdots\times\Omega_m$ and $n:=\sum_{j=1}^m n_j$. Then

(i) there exist linear operators $\mathcal{H}_q=\mathcal{H}_q^\Omega: C^\infty(\overline{\Omega};\wedge^{0,q})\to\mathscr{D}'(\Omega;\wedge^{0,q-1})$ for $1\leq q\leq n$, such that

$$f=\overline{\partial}\mathcal{H}_qf+\mathcal{H}_{q+1}\overline{\partial}f\quad\text{for all}\;\; f\in C^\infty(\overline{\Omega};\wedge^{0,q}). \tag{2}$$

Further, set the skew Bergman projections $P^{\Omega_j}:C^\infty(\overline{\Omega_j})\to\mathscr{D}'(\Omega_j)$ *for* $1\le j\le m$ *by* $P^{\Omega_j}f:=f-H_1^{\Omega_j}\overline{\partial}f$ *for* $f\in C^\infty(\overline{\Omega_j})$*, and* $\mathcal{P}=\mathcal{P}^\Omega:C^\infty(\overline{\Omega})\to\mathscr{D}'(\Omega)$ *by* $\mathcal{P}f:=f-\mathcal{H}_1^\Omega\overline{\partial}f$ *for* $f\in C^\infty(\overline{\Omega})$*.*

(ii) *Suppose there exists some* $k\in\mathbb{Z}$ *and* $1<p<\infty$*, such that for* $1\le j\le m$ *and* $1\le q\le n_j$*,* P^{Ω_j} *and* $H_q^{\Omega_j}$ *are both defined and bounded*

$$P^{\Omega_j}:W^{l,p}(\Omega_j)\to W^{l,p}(\Omega_j);\quad H_q^{\Omega_j}:W^{l,p}(\Omega_j;\wedge^{0,q})\to W^{l,p}(\Omega_j;\wedge^{0,q-1})\qquad \text{for } l=0,k. \tag{3}$$

Then $\mathcal{P}$ *and* $\mathcal{H}_q$ $(1\le q\le n)$ *obtained in (i) admit Sobolev estimates* $\mathcal{P}:W^{k,p}(\Omega)\to W^{k,p}(\Omega)$ *and* $\mathcal{H}_q:W^{k,p}(\Omega;\wedge^{0,q})\to W^{k,p}(\Omega;\wedge^{0,q-1})$ *as well.*

For the precise formulation of $\mathcal{H}^\Omega$ using $(H^{\Omega_j})_{j=1}^m$, see Remarks 20 and 23.

2. Sobolev Spaces and Fubini Property

In this section, we give the precise definition for the function space $W^{k,p}$, and discuss a Fubini property for Sobolev norms on product domains.

Notation 4. Let $\mathscr{S}'(\mathbb{R}^N)$ be the space of tempered distributions. For a bounded open subset $U\subset\mathbb{R}^N$, we denote by $\mathscr{D}'(U)$ the space of distributions in U, by $\mathscr{S}'(U)=\{\tilde f|_U:\tilde f\in\mathscr{S}'(\mathbb{R}^N)\}$ the space of extendable distributions in U, and by $\mathscr{E}'(U)$ the space of distributions with compact supports in U.

See [Ryc99, (3.1) and Proposition 3.1] for an equivalent description of $\mathscr{S}'(U)$. See also [Yao24b, Lemma A.13 (ii)].

Definition 5. Let $U\subseteq\mathbb{R}^N$ be an open subset, $k\in\mathbb{Z}_{\ge0}$, and $1\le p\le\infty$. $W^{k,p}(U)$ is the standard Sobolev space with norm

$$\|f\|_{W^{k,p}(U)}=\Big(\sum_{|\alpha|\le k}\|D^\alpha f\|_{L^p(U)}^p\Big)^{1/p}.$$

We denote by $W^{-k,p}(U):=\{\sum_{|\alpha|\le k}D^\alpha g_\alpha:g_\alpha\in L^p(U)\}$, a subset of distributions, with norm

$$\|f\|_{W^{-k,p}(U)}=\inf\Big\{\Big(\sum_{|\alpha|\le k}\|g_\alpha\|_{L^p(U)}^p\Big)^{1/p}:f=\sum_{|\alpha|\le k}D^\alpha g_\alpha\text{ as distributions}\Big\}. \tag{4}$$

Here when $p=\infty$ we take the usual modification by replacing the ℓ^p sum by the supremum.

For $l\in\mathbb{Z}$, we use $W_c^{l,p}(U)\subset W^{l,p}(U)$ to be the subspace of all functions in $W^{l,p}(U)$ that have compact supports in U.

Remark 6. (i) For $k\ge0$ and $1\le p<\infty$, let $W_0^{k,p}(U)\subseteq W^{k,p}(U)$ be the closure of $C_c^\infty(U)$ in $\|\cdot\|_{W^{k,p}(U)}$. Then we have correspondence $W^{-k,p}(U)=W_0^{k,p'}(U)'$ with equivalent norms, where $U\subset\mathbb{R}^N$ is an arbitrary domain and $p'=p/(p-1)$. See e.g. [AF03, Theorem 3.12]. It is also worth noticing that $W^{-k,p}(\mathbb{R}^N)=W^{k,p'}(\mathbb{R}^N)'$ via e.g. [AF03, Corollary 3.23].

(ii) When U is a bounded Lipschitz domain we have $\mathscr{S}'(U)=\bigcup_{k=1}^\infty W^{-k,p}(U)$, see e.g. [Yao24b, Lemma A.13 (ii)]. In other words if we have an operator T defined on $W^{-k,p}(U)$ for all $k\ge1$, then T is defined on $\mathscr{S}'(U)$.

In order to incorporate the proof of Propositions 24, 26 and 31 we include the discussion of the fractional Sobolev spaces as well.

Definition 7 (Sobolev-Bessel)**.** Let $s\in\mathbb{R}$ and $1<p<\infty$. We define the Bessel potential space $H^{s,p}(\mathbb{R}^N)$ to be the set of all tempered distributions $f\in\mathscr{S}'(\mathbb{R}^N)$ such that

$$\|f\|_{H^{s,p}(\mathbb{R}^N)}:=\|(I-\Delta)^{\frac{s}{2}}f\|_{L^p(\mathbb{R}^N)}<\infty.$$

On an open subset $U\subseteq\mathbb{R}^N$, define $H^{s,p}(U):=\{\tilde f|_U:\tilde f\in H^{s,p}(\mathbb{R}^N)\}$, with norm

$$\|f\|_{H^{s,p}(U)}:=\inf\big\{\|\tilde f\|_{H^{s,p}(\mathbb{R}^N)}:\tilde f\in H^{s,p}(\mathbb{R}^N),\ \tilde f|_U=f\big\}.$$

We also define $\widetilde H^{s,p}(\overline U):=\{f\in H^{s,p}(\mathbb{R}^N):f|_{\overline U^c}=0\}$ as a closed subspace of $H^{s,p}(\mathbb{R}^N)$.

Here we use the standard (negative) Laplacian $\Delta = \sum_{j=1}^N D_{x_j}^2$. The fractional Laplacian (Bessel potential) can be defined via Fourier transform $((I-\Delta)^{s/2}f)^\wedge(\xi) = (1+4\pi^2|\xi|^2)^{s/2}\hat{f}(\xi)$, where $\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot\xi}d\xi$.

Remark 8. Let $U \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain and $1 < p < \infty$.

(i) $H^{k,p}(U) = W^{k,p}(U)$ for all $k \in \mathbb{Z}$ with equivalent norms. See e.g. [Yao24b, Lemma A.11] for a proof.

(ii) The complex interpolation $[H^{s_0,p}(U), H^{s_1,p}(U)]_\theta = H^{(1-\theta)s_0+\theta s_1,p}(U)$ holds for all $s_0, s_1 \in \mathbb{R}$ and $0 < \theta < 1$. See e.g. [Tri06, (1.372)], where in the reference $H^{s,p}(U) = \mathscr{F}^s_{p2}(U)$ are special case of Triebel-Lizorkin spaces. As a result our homotopy operators in Theorem 1 are in fact bounded $H^{s,p}(U) \to H^{s,p}(U)$ for all $s \in \mathbb{R}$ and $1 < p < \infty$.

Lemma 9. *Let $U \subset \mathbb{R}^N$ be a bounded Lipschitz domain. There is an extension operator $E : \mathscr{S}'(U) \to \mathscr{E}'(\mathbb{R}^N)$ such that $E : W^{k,p}(U) \to W^{k,p}_c(\mathbb{R}^N)$ is bounded for all $k \in \mathbb{Z}$ and $1 < p < \infty$. In particular for each k and $1 < p < \infty$ we have $W^{k,p}(U) = \{\tilde{f}|_U : \tilde{f} \in W^{k,p}(\mathbb{R}^N)\}$.*

Proof. An existence of such extension operator E is established by Rychkov [Ryc99, Theorem 4.1]. In the reference we use the fact that $W^{k,p}(U) = \mathscr{F}^k_{p2}(U)$ are Triebel-Lizorkin spaces (see e.g. [Yao24b, Lemma A.11 (ii)]). □

Let us recall the *Fubini property* for Sobolev norms on product domains. This will enable a more convenient derivation of the Sobolev estimates for the homotopy operators. Throughout the rest of the paper, we say two quantities a and b to satisfy $a \lesssim b$ if there exists some constant C such that $a \le Cb$. We say $a \approx b$ if $a \lesssim b$ and $b \lesssim a$ at the same time.

Proposition 10 (Fubini Property)**.** *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be two bounded Lipschitz domains. Let $k \in \mathbb{Z}_+$ and $1 < p < \infty$. Then*

(i) $W^{k,p}(U \times V) = L^p(U; W^{k,p}(V)) \cap L^p(V; W^{k,p}(U))$ in the sense that we have equivalent norms

$$\|f\|^p_{W^{k,p}(U\times V)} \approx_{U,V,k,p} \int_V \|f(\cdot,v)\|^p_{W^{k,p}(U)}dv + \int_U \|f(u,\cdot)\|^p_{W^{k,p}(V)}du,$$

provided either side is finite.

(ii) $W^{-k,p}(U \times V) = L^p(U; W^{-k,p}(V)) + L^p(V; W^{-k,p}(U))$ in the sense that we have equivalent norms

$$\|f\|^p_{W^{-k,p}(U\times V)} \approx_{U,V,k,p} \inf_{f_1+f_2=f} \int_U \|f_1(u,\cdot)\|^p_{W^{-k,p}(V)}du + \int_V \|f_2(\cdot,v)\|^p_{W^{-k,p}(U)}dv,$$

provided either side is finite.

Remark 11. Here $L^p(U; X)$ can be interpreted as the space of strongly measurable functions which take values in a Banach space X, see e.g. [HvNVW16, Section 1.2.b] for more discussion.

Essentially, Proposition 10 shows that for the Sobolev functions on the product of two domains, the L^p norm of the mixed derivatives across the two domains can be controlled by the L^p norm of the pure derivatives in each individual domain.

Proof. When U and V are the total spaces $\mathbb{R}^m_u$ and $\mathbb{R}^n_v$, respectively, the decomposition

$$W^{k,p}(\mathbb{R}^m_u \times \mathbb{R}^n_v) = L^p(\mathbb{R}^m_u; W^{k,p}(\mathbb{R}^n_v)) \cap L^p(\mathbb{R}^n_v; W^{k,p}(\mathbb{R}^m_u)), \qquad k \ge 1, \quad 1 < p < \infty \tag{5}$$

is a standard result, see e.g. [Tri10, Chapter 2.5.13]. Recall from Remark 6 (i) $W^{-k,p}(\mathbb{R}^r) = W^{k,p'}(\mathbb{R}^r)'$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $r \in \{m, n, m+n\}$, taking duality (see also [HvNVW16, Proposition 1.3.3]) we get

$$W^{-k,p}(\mathbb{R}^m_u \times \mathbb{R}^n_v) = L^p(\mathbb{R}^m_u; W^{-k,p}(\mathbb{R}^n_v)) + L^p(\mathbb{R}^n_v; W^{-k,p}(\mathbb{R}^m_u)) \tag{6}$$

with equivalent norms.

(i): Now $k \geq 0$. Clearly $W^{k,p}(U\times V)\subset L^p(U;W^{k,p}(V))$ and $W^{k,p}(U\times V)\subset L^p(V;W^{k,p}(U))$, both of which are continuous embeddings. This gives $W^{k,p}(U\times V)\subseteq L^p(U;W^{k,p}(V))\cap L^p(V;W^{k,p}(U))$.

Conversely, let $E^U:\mathscr{S}'(U)\to\mathscr{S}'(\mathbb{R}^m)$ and $E^V:\mathscr{S}'(V)\to\mathscr{S}'(\mathbb{R}^n)$ be the extension operators given in Lemma 9. Define $\mathcal{E}^{U\times V}:=E^U\otimes E^V$ such that for $f(u,v)=g(u)h(v)$ we have $(\mathcal{E}^{U\times V}f)(u,v)=(E^Ug)(u)(E^Vh)(v)$. Clearly

$$\mathcal{E}^{U\times V}=(E^U\otimes\mathrm{id}^{\mathbb{R}^n})\circ(\mathrm{id}^U\otimes E^V)=(\mathrm{id}^{\mathbb{R}^m}\otimes E^V)\circ(E^U\otimes\mathrm{id}^V),$$

where $(E^U\otimes\mathrm{id}^V)f(u,v)=E^U(f(\cdot,v))(u)$ for $(u,v)\in\mathbb{R}^m\times V$ and similarly for the rest. Therefore we have the following boundedness

$$\mathcal{E}^{U\times V}:L^p(U;W^{k,p}(V))\xrightarrow{\mathrm{id}^U\otimes E^V}L^p(U;W^{k,p}(\mathbb{R}^n))\xrightarrow{E^U\otimes\mathrm{id}^{\mathbb{R}^n}}L^p(\mathbb{R}^m;W^{k,p}(\mathbb{R}^n)).$$

Similarly $\mathcal{E}^{U\times V}:L^p(V;W^{k,p}(U))\to L^p(\mathbb{R}^n;W^{k,p}(\mathbb{R}^m))$ as well.

Therefore by (5), for every $f\in W^{k,p}(U\times V)$,

$$\begin{aligned}\|f\|_{W^{k,p}(U\times V)}\leq&\|\mathcal{E}^{U\times V}f\|_{W^{k,p}(\mathbb{R}^m\times\mathbb{R}^n)}\approx\|\mathcal{E}^{U\times V}f\|_{L^p(\mathbb{R}^m;W^{k,p}(\mathbb{R}^n))}+\|\mathcal{E}^{U\times V}f\|_{L^p(\mathbb{R}^n;W^{k,p}(\mathbb{R}^m))}\\\lesssim&\|f\|_{L^p(U;W^{k,p}(V))}+\|f\|_{L^p(V;W^{k,p}(U))}.\end{aligned}$$

We conclude that (i) holds.

(ii): Clearly $L^p(U;W^{-k,p}(V))\subset W^{-k,p}(U\times V)$ and $L^p(V;W^{-k,p}(U))\subset W^{-k,p}(U\times V)$ by (4). This gives the embedding $L^p(U;W^{-k,p}(V))+L^p(V;W^{-k,p}(U))\subseteq W^{-k,p}(U\times V)$.

Conversely, for every $f\in W^{-k,p}(U\times V)$, by Lemma 9 it admits an extension $\tilde f\in W^{-k,p}(\mathbb{R}^m\times\mathbb{R}^n)$. By (6) there exist $\tilde f_1\in L^p(\mathbb{R}^m_u;W^{-k,p}(\mathbb{R}^n_v))$ and $\tilde f_2\in L^p(\mathbb{R}^n_v;W^{-k,p}(\mathbb{R}^m_u))$ such that $\tilde f=\tilde f_1+\tilde f_2$. Taking restrictions we get the existence of the decomposition $f=f_1+f_2$ where $f_1:=\tilde f_1|_{U\times V}\in L^p(U;W^{-k,p}(V))$ and $f_2:=\tilde f_2\in L^p(V;W^{-k,p}(U))$.

Now for given $f\in W^{-k,p}(U\times V)$, let $f_1\in L^p(U;W^{-k,p}(V))$ and $f_2\in L^p(V;W^{-k,p}(U))$ be arbitrary functions such that $f_1+f_2=f$, which exist from above. Thus

$$\|f\|_{W^{-k,p}(U\times V)}=\|f_1\|_{W^{-k,p}(U\times V)}+\|f_2\|_{W^{-k,p}(U\times V)}\lesssim\|f_1\|_{L^p(U;W^{-k,p}(V))}+\|f_2\|_{L^p(V;W^{-k,p}(U))}.$$

Therefore by taking infimum over the decomposition $f_1+f_2=f$ we conclude that

$$\|f\|^p_{W^{-k,p}(U\times V)}\lesssim\inf_{f_1+f_2=f}\int_U\|f_1(u,\cdot)\|^p_{W^{-k,p}(V)}du+\int_V\|f_2(\cdot,v)\|^p_{W^{-k,p}(U)}dv.$$

That is, $W^{-k,p}(U\times V)\subseteq L^p(U;W^{-k,p}(V))+L^p(V;W^{-k,p}(U))$, completing the proof of (ii). ☐

Corollary 12. *Let $U\subset\mathbb{R}^m,V\subset\mathbb{R}^n$ be two bounded Lipschitz domains, and $k\in\mathbb{Z}$, $1<p<\infty$. Let $T:L^p(U)\to L^p(U)$ be a bounded linear operator that extends to a bounded linear map $T:W^{k,p}(U)\to W^{k,p}(U)$. Defines $\mathcal{T}:L^p(U\times V)\to L^p(U\times V)$ by acting T on the coordinate component of U, i.e. $\mathcal{T}f(u,v):=T(f(\cdot,v))(u)$. Then $\mathcal{T}:W^{k,p}(U\times V)\to W^{k,p}(U\times V)$ is defined and bounded.*

Here we are indeed using $\mathcal{T}=T\otimes\mathrm{id}^V$. See also Convention 17.

Proof. The definition $\mathcal{T}D_v=D_v\mathcal{T}$ and the property $L^p(U;L^p(V))=L^p(V;L^p(U))$ ensure that $\mathcal{T}:L^p(U;W^{k,p}(V))\to L^p(U;W^{k,p}(V))$. The boundedness $\mathcal{T}:L^p(V;W^{k,p}(U))\to L^p(V;W^{k,p}(U))$ is a direct consequence of that of $T:W^{k,p}(U)\to W^{k,p}(U)$. The $W^{k,p}(U\times V)\to W^{k,p}(U\times V)$ bound of $\mathcal{T}$ then follows from Proposition 10. ☐

3. Nijenhuis-Woolf Formulae and the proof of Theorem 3

In this section, we shall construct homotopy formulae on product domains making use of an idea in [NW63]. This together with Proposition 10 allows us to prove Theorem 3.

First we introduce some notations and conventions for linear operators defined on forms (of mixed degrees), which will be used to facilitate our proof.

Convention 13 (Spaces on forms)**.** Let $\mathscr{X}\in\{\mathscr{S}',\mathscr{D}',C^\infty,W^{k,p},H^{s,p}:k\in\mathbb{Z},s\in\mathbb{R},1<p<\infty\}$ and let $U\subseteq\mathbb{C}^n$. For $1\le q\le n$ we use $\mathscr{X}(U;\wedge^{0,q})$ the space of $(0,q)$ forms $f(\zeta)=\sum_{|I|=q}f_I(\zeta)d\bar\zeta^I$ where $f_I\in\mathscr{X}(U)$ for all I. If $\mathscr{X}\in\{W^{k,p},H^{s,p}\}$, then we use the norm $\|f\|_{\mathscr{X}(U;\wedge^{0,q})}=\sum_{|I|=q}\|f_I\|_{\mathscr{X}(U)}$. Denote by $\mathscr{X}(U;\wedge^{0,\bullet})=\bigoplus_{q=0}^n\mathscr{X}(U;\wedge^{0,q})$ for forms of mixed degrees.

We adopt the following convention to extend an operator originally defined on forms of a single degree to one on forms of mixed degrees.

Convention 14 (Operators on mixed degree forms)**.** Let $U\subseteq\mathbb{C}^n$, $0\le q,r\le n$ and $\mathscr{X},\mathscr{Y}\in\{\mathscr{S}',\mathscr{D}',C^\infty,W^{k,p},H^{s,p}\}$. We identify a linear operator $S:\mathscr{X}(U;\wedge^{0,q})\to\mathscr{Y}(U;\wedge^{0,r})$ as $S:\mathscr{X}(U;\wedge^{0,\bullet})\to\mathscr{Y}(U;\wedge^{0,\bullet})$ by setting $S(f_Jd\bar\zeta_J)=0$ if $|J|\ne q$.

For a family of operators $(T_q)_{q=0}^n$ where each T_q is defined on $(0,q)$ forms, we use $T=\sum_{q=0}^nT_q$ to denote the corresponding operator on mixed degree forms. Namely, for a form $f(z)=\sum_{q=0}^nf_q(z)$ where $f_q(z)$ is of degree $(0,q)$, we define $Tf=\sum_{q=0}^nT_qf_q$.

Remark 15. Under this convention, we can rewrite the homotopy formulae in Theorems 1 and 3 as a single formula (here we use $\mathcal{H}_0=\mathcal{H}_{n+1}=0$)

$$f=\mathcal{P}f+\overline\partial\mathcal{H}f+\mathcal{H}\overline\partial f,\quad\text{for}\quad f\in\mathscr{S}'(\Omega;\wedge^{0,\bullet}),\qquad\text{where}\quad\mathcal{H}=\textstyle\sum_{q\ge1}\mathcal{H}_q.$$

Remark 16. For an operator S defined on functions, namely, with $q=r=0$ in Convention 14, one extends S on differential forms by taking zero value on $(0,q)$ forms when $q\ge1$. In the paper not all the operators on functions follow this convention. For example for an extension operator $E:\mathscr{X}(U)\to\mathscr{X}(\mathbb{C}^n)$ we define it on forms by acting on components, i.e. $(Ef)(z)=\sum_I(Ef_I)(z)d\bar z^I$.

Next, we extend operators originally defined on slices to the entire product domain using the following convention.

Convention 17 (Operator on product domains)**.** Let $U\subseteq\mathbb{C}^m$ and $V\subseteq\mathbb{C}^n$ be two open sets, endowed with standard coordinate system $z=(z^1,\dots,z^m)$ and $\zeta=(\zeta^1,\dots,\zeta^n)$ respectively. For a linear operator $T^U:\mathscr{X}(U;\wedge^{0,\bullet})\to\mathscr{Y}(U;\wedge^{0,\bullet})$, we denote $\mathcal{T}^U$ for the associated operator $T^U\otimes\mathrm{id}^V$ on $(0,\bullet)$ forms defined on $U\times V$ by setting

$$\mathcal{T}^U(\omega\wedge d\bar\zeta^K)(z,\zeta):=T^U(\omega(\cdot,\zeta))(z)\wedge d\bar\zeta^K,\quad\text{where }\omega(z,\zeta)=\sum_I\omega_I(z,\zeta)d\bar z^I.\tag{7}$$

In particular, if we write $T^U(\sum_Jg_Jd\bar z^J)=:\sum_{I,J}(T^U_{IJ}g_I)d\bar z^J$ where $T^U_{IJ}:\mathscr{X}(U)\to\mathscr{Y}(U)$ are linear operators on functions, then for a form $f(z,\zeta)=\sum_{J,K}f_{JK}(z,\zeta)d\bar z^J\wedge d\bar\zeta^K$,

$$(\mathcal{T}^Uf)(z,\zeta)=\sum_{I,J,K}\{T^U_{IJ}[f_{IK}(\cdot,\zeta)]\}(z)d\bar z^I\wedge d\bar\zeta^K,\quad z\in U,\quad\zeta\in V.$$

Motivated by a one-dimensional analogue in [NW63, (2.2.2) - (2.2.5)], we deduce our homotopy formulae making use of the following product-type configuration. See also Remark 20.

Theorem 18 (Product homotopy formulae)**.** *Let $U\subset\mathbb{C}^{n_U}$ and $V\subset\mathbb{C}^{n_V}$ be two open subsets. Suppose for each $W\in\{U,V\}$, there exist continuous linear operators $H^W_q:C^\infty(\overline W;\wedge^{0,q})\to\mathscr{D}'(W;\wedge^{0,q-1})$ for $1\le q\le n_W$, such that the following homotopy formulae hold ($H^W_{n_W+1}=0$ as usual) for $1\le q\le n_W$:*

$$f=\overline\partial H^W_qf+H^W_{q+1}\overline\partial f\quad\textit{for all}\quad f\in C^\infty(\overline W;\wedge^{0,q}).\tag{8}$$

Set $P^Wf:=f-H^W_1\overline\partial f$ for functions $f\in C^\infty(\overline W)$.

(i) Then we have homotopy formulae $f=\mathcal{P}^{U\times V}f+\overline\partial\mathcal{H}^{U\times V}f+\mathcal{H}^{U\times V}\overline\partial f$ for $f\in C^\infty(\overline{U\times V};\wedge^{0,\bullet})$, where (see Conventions 14 and 17)

$$\mathcal{P}^{U\times V}:=\mathcal{P}^U\circ\mathcal{P}^V=P^U\otimes P^V;\qquad\mathcal{H}^{U\times V}:=\mathcal{H}^U+\mathcal{P}^U\circ\mathcal{H}^V=H^U\otimes\mathrm{id}^V+P^U\otimes H^V.\tag{9}$$

(ii) Let $k \in \mathbb{Z}$ and $1 < p < \infty$. Suppose further that U and V are bounded Lipschitz domains, $P^U, H^U : W^{l,p}(U;\wedge^{0,\bullet}) \to W^{l,p}(U;\wedge^{0,\bullet})$ and $H^V : W^{l,p}(V;\wedge^{0,\bullet}) \to W^{l,p}(V;\wedge^{0,\bullet})$ are all defined and bounded for $l \in \{0,k\}$, then $\mathcal{H}^{U\times V}$ given in (9) are defined and bounded in $W^{k,p}(U\times V;\wedge^{0,\bullet}) \to W^{k,p}(U\times V;\wedge^{0,\bullet})$ as well.

If in addition $P^V : W^{l,p}(V) \to W^{l,p}(V)$ is bounded for $l \in \{0,k\}$, then $\mathcal{P}^{U\times V} : W^{k,p}(U\times V) \to W^{k,p}(U\times V)$ is bounded as well.

Remark 19. It is worth pointing out that the original formulae of Nijenhuis-Woolf in [NW63] are restricted to products of planar domains. Under their settings the $\mathcal{H}^{\Omega_j}$ operators (denoted by T^j in the reference) are the solid Cauchy integral over Ω_j. In their notation the operators are defined on functions rather than $(0,1)$ forms. Moreover, even in the case when each Ω_j is smooth, while $\mathcal{H}^{\Omega_j}$ there satisfies the desired $W^{k,p}$ regularity for $k \geq 0$ (see, for instance, [PZ25, Proposition 3.1] with $\mu \equiv 1$ there), their $\mathcal{P}^{\Omega_j}$ operators (denoted by S^j in the reference), which are given by the boundary Cauchy integrals over $b\Omega_j$, do not yield well-defined or bounded mappings on the L^p space as required in Theorem 18 (ii). In Proposition 24 in the next section, we shall introduce a slightly different choice of homotopy operators to overcome this issue.

Remark 20 (Formulae with separated degrees). Let $z = (z^1, \dots, z^{n_U})$ and $\zeta = (\zeta^1, \dots, \zeta^{n_V})$ be standard coordinate systems for $\mathbb{C}^{n_U}$ and $\mathbb{C}^{n_V}$, respectively. For $0 \leq j \leq n_U$ and $0 \leq k \leq n_V$, let us define the standard projection $\pi_{j,k}$ of forms by

$$\pi_{j,k} f := \sum_{|J|=j,|K|=k} f_{JK} d\bar{z}^J \wedge d\bar{\zeta}^K, \quad \text{for every } f = \sum_{J\subseteq\{1,\dots,n_U\},K\subseteq\{1,\dots,n_V\}} f_{JK} d\bar{z}^J \wedge d\bar{\zeta}^K.$$

We have $f = \sum_{j=0}^{n_U}\sum_{k=0}^{n_V} \pi_{j,k} f$ and $\sum_{j+k=q} \pi_{j,k} f$ is the degree $(0,q)$ components of f.

Under this notation, and Conventions 14 and 17, we can write $\mathcal{H}_q^{U\times V}$ in (9) for $1 \leq q \leq n_U + n_V$ as

$$\begin{aligned}\mathcal{H}_q^{U\times V} &= \mathcal{H}^U \circ \Big(\sum_{j+k=q} \pi_{j,k}\Big) + \mathcal{P}^U \circ \mathcal{H}^V \circ \Big(\sum_{j+k=q} \pi_{j,k}\Big) \\ &= \sum_{j=1}^{q} \mathcal{H}_j^U \circ \pi_{j,q-j} + \mathcal{P}^U \circ \mathcal{H}_q^V \circ \pi_{0,q} = \sum_{j=1}^{q} (H_j^U \otimes \mathrm{id}^V) \circ \pi_{j,q-j} + (P^U \otimes H_q^V) \circ \pi_{0,q}.\end{aligned}$$

Note that the formula (9) is asymmetric. Namely, if we swap U and V, the homotopy operators in (9) are not the same.

Remark 21. It is important that U and V in the assumption are at most Lipschitz. In the proof of Theorem 1 and 3 we use induction with $U = \Omega_1 \times \cdots \times \Omega_{m-1}$ and $V = \Omega_m$. However, even for two smooth domains, the boundary regularity of their product is merely Lipschitz.

In contrast, Theorem 18 (ii) remains true for non-Lipschitz domains U and V if the analogy of Proposition 10 holds for such U and V. A typical example is the Hartogs triangle, which is a non-Lipschitz but uniform domain due to [BFLS22].

To prove Theorem 18, we let $z = (z^1, \dots, z^{n_U})$ and $\zeta = (\zeta^1, \dots, \zeta^{n_V})$ be the standard coordinate systems for $\mathbb{C}^{n_U}$ and $\mathbb{C}^{n_V}$, respectively. We denote by $\overline{\partial}_z = \sum_{j=1}^{n_U} \overline{\partial} z^j \wedge \frac{\partial}{\partial \bar{z}^j}$ and $\overline{\partial}_\zeta = \sum_{k=1}^{n_V} \overline{\partial}\zeta^k \wedge \frac{\partial}{\partial \bar{\zeta}^k}$ the $\overline{\partial}$-operators of the z-component and the ζ-component, respective. Note that on the product domain $U \times V$ we have $\overline{\partial} = \overline{\partial}_{z,\zeta} := \overline{\partial}_z + \overline{\partial}_\zeta$.

The key computation is the following (cf. [NW63, (2.1.1) - (2.1.3)]):

Lemma 22. *Keeping the notations as above and as in Theorem 18, on $U \times V$ we have*

$$\overline{\partial}_{z,\zeta}\mathcal{P}^U = \mathcal{P}^U \overline{\partial}_{z,\zeta}, \quad \overline{\partial}_{z,\zeta}\mathcal{P}^V = \mathcal{P}^V \overline{\partial}_{z,\zeta}, \quad \overline{\partial}_\zeta \mathcal{H}^U = -\mathcal{H}^U \overline{\partial}_\zeta, \quad \overline{\partial}_z \mathcal{H}^V = -\mathcal{H}^V \overline{\partial}_z. \tag{10}$$

Moreover,

$$\overline{\partial}\mathcal{H}^U + \mathcal{H}^U\overline{\partial} = \mathrm{id} - \mathcal{P}^U, \qquad \overline{\partial}\mathcal{H}^V + \mathcal{H}^V\overline{\partial} = \mathrm{id} - \mathcal{P}^V, \tag{11}$$

provided that the operators on both sides of the equalities are defined.

Proof. Recall that $P^U=\mathrm{id}_0-H_1^U\overline{\partial}_z$ is a projection for functions on U to holomorphic functions on U. Therefore $\overline{\partial}_zP^U=0$. Since P^U vanishes on $(0,q)$ forms when $q\ge1$, we get $P^U\overline{\partial}_z=0$. Together by Convention 17 we have $\overline{\partial}_z\mathcal{P}^U=\mathcal{P}^U\overline{\partial}_z=0$. Since $\mathcal{P}^U$ only acts on z-variable, using (7) as well we get $\overline{\partial}_\zeta\mathcal{P}^U=\mathcal{P}^U\overline{\partial}_\zeta$. Together we have $\overline{\partial}_{z,\zeta}\mathcal{P}^U=\mathcal{P}^U\overline{\partial}_{z,\zeta}$. The same argument yields $\overline{\partial}_{z,\zeta}\mathcal{P}^V=\mathcal{P}^V\overline{\partial}_{z,\zeta}$.

Next, for a form $f(z,\zeta)=f_{JK}(z,\zeta)d\bar z^J\wedge d\bar\zeta^K$, the $\mathcal{H}^U(f_{JK}d\bar z^J)$ is a $(0,|J|-1)$ form. By a direct computation and (7),

$$\begin{aligned}&\overline{\partial}_\zeta\mathcal{H}^Uf=\overline{\partial}_\zeta\mathcal{H}^U(f_{JK}d\bar z^J\wedge d\bar\zeta^K)=\sum_{k=1}^{n_V}d\bar\zeta^k\wedge\frac{\partial}{\partial\bar\zeta^k}\mathcal{H}^U(f_{JK}d\bar z^J)\wedge d\bar\zeta^K\\
=&(-1)^{|J|-1}\sum_{k=1}^{n_V}\frac{\partial}{\partial\bar\zeta^k}\mathcal{H}^U(f_{JK}d\bar z^J)\wedge d\bar\zeta^k\wedge d\bar\zeta^K=(-1)^{|J|-1}\sum_{k=1}^{n_V}\mathcal{H}^U\Big(\frac{\partial f_{JK}}{\partial\bar\zeta^k}d\bar z^J\wedge d\bar\zeta^k\wedge d\bar\zeta^K\Big)\\
=&(-1)^{|J|-1}(-1)^{|J|}\sum_{k=1}^{n_V}\mathcal{H}^U\Big(d\bar\zeta^k\wedge\frac{\partial f_{JK}}{\partial\bar\zeta^k}d\bar z^J\wedge d\bar\zeta^K\Big)=-\mathcal{H}^U\overline{\partial}_\zeta(f_{JK}d\bar z^J\wedge d\bar\zeta^K)=-\mathcal{H}^U\overline{\partial}_\zeta f.\end{aligned}$$

We get $\overline{\partial}_\zeta\mathcal{H}^U=-\mathcal{H}^U\overline{\partial}_\zeta$. By swapping (z,U) and (ζ,V) we get $\overline{\partial}_z\mathcal{H}^V=\mathcal{H}^V\overline{\partial}_z$. This completes the proof of (10).

Since by assumption $\mathrm{id}^U-P^U=\overline{\partial}_zH^U+H^U\overline{\partial}_z$ and $\mathrm{id}^V-P^V=\overline{\partial}_\zeta H^V+H^V\overline{\partial}_\zeta$. Combing them with (10) and Convention 17, we have (11). □

Proof of Theorem 18. First we note that $\mathcal{P}^{U\times V},\mathcal{H}^{U\times V}$ from (9) are always defined on $C^\infty(\overline{U\times V};\wedge^{0,\bullet})$. Indeed, for $W\in\{U,V\}$, a continuous linear operator $T^W:C^\infty(\overline{W})\to\mathscr{D}'(W)$ can be lifted as a linear operator $\widetilde{T}^W:C_c^\infty(\mathbb{C}^{n_W})\to\mathscr{D}'(W)$ via a continuous extension operator $C^\infty(\overline{W})\to C_c^\infty(\mathbb{C}^{n_W})$. By the Schwartz Kernel Theorem (see e.g. [Trè06, Theorem 51.7]) $\widetilde{T}^U\otimes\widetilde{T}^V:C_c^\infty(\mathbb{C}^{n_U}\times\mathbb{C}^{n_V})\to\mathscr{D}'(U\times V)$ is continuous. Clearly $((\widetilde{T}^U\otimes\widetilde{T}^V)\tilde f)|_{U\times V}=(T^U\otimes T^V)(\tilde f|_{U\times V})$ for all $\tilde f\in C_c^\infty(\mathbb{C}^{n_U+n_V})$, we get the continuity $T^U\otimes T^V:C^\infty(\overline{U\times V})\to\mathscr{D}'(U\times V)$. Take $T^U\in\{P^U,H^U\}$ and $T^V\in\{P^V,H^V\}$ we get the definedness.

Using (11) for every $f\in C^\infty(\overline{U\times V};\wedge^{0,\bullet})$,

$$\begin{aligned}&\mathcal{P}^{U\times V}f+\overline{\partial}\mathcal{H}^{U\times V}f+\mathcal{H}^{U\times V}\overline{\partial}f=\mathcal{P}^U\mathcal{P}^Vf+\overline{\partial}(\mathcal{H}^U+\mathcal{P}^U\mathcal{H}^V)f+(\mathcal{H}^U+\mathcal{P}^U\mathcal{H}^V)\overline{\partial}f\\
=&\mathcal{P}^U\mathcal{P}^Vf+(\overline{\partial}\mathcal{H}^U+\mathcal{H}^U\overline{\partial})f+(\overline{\partial}\mathcal{P}^U\mathcal{H}^V+\mathcal{P}^U\mathcal{H}^V\overline{\partial})f=\mathcal{P}^U\mathcal{P}^Vf+f-\mathcal{P}^Uf+\mathcal{P}^U(\overline{\partial}\mathcal{H}^V+\mathcal{H}^V\overline{\partial})f\\
=&\mathcal{P}^U\mathcal{P}^Vf+f-\mathcal{P}^Uf+\mathcal{P}^Uf-\mathcal{P}^U\mathcal{P}^Vf=f.\end{aligned}$$

This gives the proof of (i).

For (ii), by Corollary 12 the boundedness assumption of $P^U,H^U:W_z^{k,p}(U;\wedge^{0,\bullet})\to W_z^{k,p}(U;\wedge^{0,\bullet})$ and $P^U,H^U:L_z^p(U;\wedge^{0,\bullet})\to L_z^p(U;\wedge^{0,\bullet})$ implies the boundedness of $\mathcal{P}^U,\mathcal{H}^U:W_{z,\zeta}^{k,p}(U\times V;\wedge^{0,\bullet})\to W_{z,\zeta}^{k,p}(U\times V;\wedge^{0,\bullet})$. The same argument yields the boundedness $\mathcal{H}^V:W_{z,\zeta}^{k,p}(U\times V;\wedge^{0,\bullet})\to W_{z,\zeta}^{k,p}(U\times V;\wedge^{0,\bullet})$. By (9) with compositions, we conclude that $\mathcal{H}^{U\times V}:W_{z,\zeta}^{k,p}(U\times V;\wedge^{0,\bullet})\to W_{z,\zeta}^{k,p}(U\times V;\wedge^{0,\bullet})$ is bounded.

If we further assume P^V is bounded in both $L_\zeta^p(V)\to L_\zeta^p(V)$ and $W_\zeta^{k,p}(V)\to W_\zeta^{k,p}(V)$, then by Corollary 12 $\mathcal{P}^V:W_{z,\zeta}^{k,p}(U\times V)\to W_{z,\zeta}^{k,p}(U\times V)$ is bounded as well. Taking compositions with $\mathcal{P}^U$, we obtain the boundedness $\mathcal{P}^{U\times V}:W_{z,\zeta}^{k,p}(U\times V)\to W_{z,\zeta}^{k,p}(U\times V)$, completing the proof. □

Proof of Theorem 3. Identifying $\mathcal{P}$ as the operator on forms of all degrees following Convention 14 and Remark 15, we can write the homotopy formulae as $f=\mathcal{P}f+\overline{\partial}\mathcal{H}f+\mathcal{H}\overline{\partial}f$ for mixed degree form f.

The proof can be done by induction on m. The based case $m=1$ follows from the assumption (1). Suppose the case $m-1$ is obtained. For the case m, take $U:=\Omega_1\times\cdots\times\Omega_{m-1}\subset\mathbb{C}^{n_1+\cdots+n_{m-1}}$ and $V:=\Omega_m\subset\mathbb{C}^{n_m}$. Since the product of bounded Lipschitz domains is still bounded Lipschitz, we see that U and V are both bounded Lipschitz domains as well. By the induction hypothesis there are linear operators $H^U=\sum_{q=1}^{n_1+\dots+n_{m-1}}H^U_q$ on $C^\infty(\overline{U};\wedge^{0,\bullet})$ and $H^V=\sum_{q=1}^{n_m}H^V_q$ on $C^\infty(\overline{V};\wedge^{0,\bullet})$ (in terms of Conventions 14 and 17), such that $g=P^Ug+\overline{\partial}H^Ug+H^U\overline{\partial}g$ for all $g\in C^\infty(\overline{U};\wedge^{0,\bullet})$, $h=P^Vh+\overline{\partial}H^Vh+H^V\overline{\partial}h$ for all $h\in C^\infty(\overline{V};\wedge^{0,\bullet})$, where $P^U=\mathrm{id}^U_0-H^U_1\overline{\partial}$ and $P^V=\mathrm{id}^V_0-H^V_1\overline{\partial}$ are skew Bergman projections on functions (in U and V respectively).

Applying Theorem 18 to such P^U,H^U,P^V,H^U we obtain the desired operators $\mathcal{P}^\Omega$ and $\mathcal{H}^\Omega=(\mathcal{H}^\Omega_q)_{q=1}^{n_1+\dots+n_m}$ on $\Omega=U\times V$. By Theorem 18 (i) $f=\mathcal{P}^\Omega f+\overline{\partial}\mathcal{H}^\Omega f+\mathcal{H}^\Omega\overline{\partial}f$ for all $f\in C^\infty(\overline{\Omega};\wedge^{0,\bullet})$, which gives (2).

Suppose further (3) holds, that is, for some $k\in\mathbb{Z}$ and $1<p<\infty$, P^U,P^V,H^U,H^V are all bounded in $W^{k,p}\to W^{k,p}$ and $L^p\to L^p$. By Theorem 18 (ii) the $W^{k,p}$ and L^p boundeness for P^U,H^U,P^V,H^U implies the $W^{k,p}$ boundedness for $\mathcal{P}^\Omega$ and $\mathcal{H}^\Omega$. (ii) is thus proved. □

Remark 23. By expanding the induction, the formulae we have for $\Omega=\Omega_1\times\dots\times\Omega_r$ are

$$\begin{aligned}\mathcal{P}^\Omega=&\mathcal{P}^{\Omega_1}\dots\mathcal{P}^{\Omega_m}=P^{\Omega_1}\otimes\dots\otimes P^{\Omega_m};\\ \mathcal{H}^\Omega=&\mathcal{H}^{\Omega_1}+\mathcal{P}^{\Omega_1}\mathcal{H}^{\Omega_2}+\dots+\mathcal{P}^{\Omega_1}\dots\mathcal{P}^{\Omega_{m-1}}\mathcal{H}^{\Omega_m}\\ &=H^{\Omega_1}\otimes\mathrm{id}^{\Omega_2\times\dots\times\Omega_m}+P^{\Omega_1}\otimes H^{\Omega_2}\otimes\mathrm{id}^{\Omega_3\times\dots\times\Omega_m}+\dots+P^{\Omega_1}\otimes\dots\otimes P^{\Omega_{m-1}}\otimes H^{\Omega_m}.\end{aligned}$$

For a given degree $(0,q)$, the precise expression of $\mathcal{H}^\Omega_q$ follows from the same deduction to Remark 20.

4. Proof of Theorem 1

In this section, we check that for each factor Ω_j under consideration in Theorem 1, there exist linear operators $(H_q^{\Omega_j})_{q=1}^{n_j}$ and P which satisfy the homotopy formulae and has the desired boundedness in all Sobolev spaces.

Proposition 24. *Let $\Omega\subset\mathbb{C}$ be a bounded Lipschitz domain. Then there is an operator $H_1:\mathscr{S}'(\Omega;\wedge^{0,1})\to\mathscr{S}'(\Omega)$ such that $\overline{\partial}H_1=\mathrm{id}$ and $H_1:W^{k,p}(\Omega;\wedge^{0,1})\to W^{k+1,p}(\Omega)$ is bounded for all $k\in\mathbb{Z}$ and $1<p<\infty$. In particular $P:=\mathrm{id}-H_1\overline{\partial}$ satisfies $P:W^{k,p}(\Omega)\to W^{k,p}(\Omega)$ for all $k\in\mathbb{Z}$ and $1<p<\infty$, and we have the homotopy formula $f=Pf+\overline{\partial}Hf+H\overline{\partial}f$ for $f\in\mathscr{S}'(\Omega,\wedge^{0,\bullet})$.*

Note that since there are no $(0,2)$ forms in $\mathbb{C}^1$, we have $H=H_1$. In particular, $f=Pf+H_1\overline{\partial}f$ for functions $f\in\mathscr{S}'(\Omega)$ and $f=\overline{\partial}H_1f$ for $(0,1)$ forms $f\in\mathscr{S}'(\Omega;\wedge^{0,1})$.

Proof. Take a bounded open set $U\Supset\Omega$. By Lemma 9 there exists an extension operator $E:\mathscr{S}'(\Omega)\to\mathscr{E}'(U)$ such that $E:W^{k,p}(\Omega)\to W^{k,p}_c(U)$ is bounded for all $k\in\mathbb{Z}$ and $1<p<\infty$. Take

$$H_1(gd\bar z):=(\tfrac{1}{\pi z}*Eg)|_\Omega.$$

Since $\frac{1}{\pi z}$ is the fundamental solution to $\frac{\partial}{\partial\bar z}$, we get $\overline{\partial}H_1(gd\bar z)=gd\bar z$ for all $g\in\mathscr{S}'(\Omega)$. The boundedness $H_1:W^{k,p}(\Omega;\wedge^{0,1})\to W^{k+1,p}(\Omega)$ is standard, from which one simultaneously obtains the boundedness $P:W^{k,p}(\Omega)\to W^{k,p}(\Omega)$. We give a version of the proof here.

Since $E:W^{k,p}(\Omega)\to W^{k,p}_c(U)$ is bounded, it suffices to show the boundedness $[g\mapsto\frac{1}{\pi z}*g]:W^{k,p}_c(U)\to W^{k+1,p}(\Omega)$. Since U is bounded, say $U\subset B(0,R)$, we can take a $\chi\in C_c^\infty(\mathbb{C})$ such that $\chi|_{B(0,2R)}\equiv1$, which allows $(\frac{1}{\pi z}*g)|_\Omega=((\chi\cdot\frac{1}{\pi z})*g)|_\Omega$. Thus the proposition is further reduced to showing $[g\mapsto(\chi\cdot\frac{1}{\pi z})*g]:W^{k,p}(\mathbb{C})\to W^{k+1,p}(\mathbb{C})$ is bounded.

Recalling that for the Fourier transform $\hat f(\xi,\eta)=\int_{\mathbb{C}}f(x+iy)e^{-2\pi i(x\xi+y\eta)}dxdy$, we see that

$$m(\xi,\eta):=\Big((I-\Delta)^{\frac12}(\chi\cdot\tfrac{1}{\pi z})\Big)^\wedge(\xi,\eta)=\tfrac{1}{\pi i}\sqrt{1+4\pi^2(|\xi|^2+|\eta|^2)}\cdot\Big(\check\chi*\tfrac{1}{\xi+i\eta}\Big).$$

This is a bounded smooth function in $\mathbb{R}^2_{\xi,\eta}$ such that $\sup_{\xi,\eta}\sqrt{\xi^2+\eta^2}|\nabla m(\xi,\eta)|<\infty$, which is in particular a Hörmander-Mikhlin multiplier. By the Hörmander-Mikhlin multiplier theorem (see e.g. [Gra14, Section 6.2.3]) $[g\mapsto(I-\Delta)^{\frac12}(\chi\cdot\frac{1}{\pi z})*g]:L^p(\mathbb{C})\to L^p(\mathbb{C})$ is bounded for all $1<p<\infty$.

Using the Sobolev-Bessel spaces in Definition 7 and the fact that $(I-\Delta)^{\frac{s}{2}}(\check{m}*g)=\check{m}*(I-\Delta)^{\frac{s}{2}}g$, we conclude that $[g\mapsto(\chi\cdot\frac{1}{\pi z})*g]:H^{s,p}(\mathbb{C})\to H^{s+1,p}(\mathbb{C})$ is bounded for all $s\in\mathbb{R}$ and $1<p<\infty$. The $W^{k,p}$ boundedness follows from Remark 8 (i). □

Proposition 25. *Let $\Omega\subset\mathbb{C}^n$ be a bounded domain which is either C^2 strongly pseudoconvex or $C^{1,1}$ strongly $\mathbb{C}$-linearly convex. There are linear operators $P:\mathscr{S}'(\Omega)\to\mathscr{S}'(\Omega)$ and $H_q:\mathscr{S}'(\Omega;\wedge^{0,q})\to\mathscr{S}'(\Omega;\wedge^{0,q-1})$ for $1\le q\le n$, such that $f=Pf+\overline{\partial}Hf+H\overline{\partial}f$ for all $f\in\mathscr{S}'(\Omega,\wedge^{0,\bullet})$, and $P,H:W^{k,p}(\Omega;\wedge^{0,\bullet})\to W^{k,p}(\Omega;\wedge^{0,\bullet})$ are bounded for all $k\in\mathbb{Z}$ and $1<p<\infty$.*

See [Yao24b, Theorem 1.1]. In fact we have the boundedness $H_q:H^{s,p}(\Omega;\wedge^{0,q})\to H^{s+1/2,p}(\Omega;\wedge^{0,q-1})$ for $1\le q\le n-1$ and $H_n:H^{s,p}(\Omega;\wedge^{0,n})\to H^{s+1,p}(\Omega;\wedge^{0,n-1})$ for all $s\in\mathbb{R}$ and $1<p<\infty$.

Proposition 26. *Let $\Omega\subset\mathbb{C}^n$ be a smooth convex domain of finite type. There are linear operators $P:\mathscr{S}'(\Omega)\to\mathscr{S}'(\Omega)$ and $H_q:\mathscr{S}'(\Omega;\wedge^{0,q})\to\mathscr{S}'(\Omega;\wedge^{0,q-1})$ for $1\le q\le n$, such that $f=Pf+\overline{\partial}Hf+H\overline{\partial}f$ for all $f\in\mathscr{S}'(\Omega,\wedge^{0,\bullet})$, and $P,H:W^{k,p}(\Omega;\wedge^{0,\bullet})\to W^{k,p}(\Omega;\wedge^{0,\bullet})$ are bounded for all $k\in\mathbb{Z}$ and $1<p<\infty$.*

The boundedness of H_q was obtained in [Yao24b]. For the boundedness of $P=\mathrm{id}_0-H_1\overline{\partial}$, we postpone the proof to Theorem 29 in Section A. A slightly more general version of this statement using Triebel-Lizorkin spaces can be found in the arxiv version [Yao24a, Appendix B] with a similar argument as in the Appendix.

Theorem 1 now follows directly from Theorem 3 with Propositions 24 - 26. We include the proof for completeness.

Proof of Theorem 1 and Corollary 2. Since on each Ω_j we have (1) and (3) for all $k\in\mathbb{Z}$ and $1<p<\infty$ by Propositions 24 - 26, we obtain the linear operators $\mathcal{P}$ and $\mathcal{H}$ as defined in Theorem 3, which satisfy (2), and are bounded on $W^{k,p}$ for all $k\in\mathbb{Z}$ and $1<p<\infty$. Because $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ (see e.g. [Yao24b, Lemma A.14] for $k\le 0$), the homotopy formulae uniquely extends to all $f\in W^{k,p}(\Omega;\wedge^{0,\bullet})$ for $k\le 0$ and $1<p<\infty$. By Remark 6 (ii) again the homotopy formula (2) holds for all $f\in\mathscr{S}'(\Omega;\wedge^{0,\bullet})$. This proves Theorem 1. Corollary 2 is a direct consequence of Theorem 1. □

Remark 27. If one only focuses on optimal $W^{k,p}$ estimates for $k\ge 0$, we can also allow Ω_j in Theorem 1 to be a smooth pseudoconvex domain of finite type in $\mathbb{C}^2$, or other pseudoconvex domains where the canonical solution operators $H_q:=\overline{\partial}^*N_q$ and the Bergman projection $P:=\mathrm{id}_0-\overline{\partial}^*N_1\overline{\partial}$ are bounded in $W^{k,p}$. See e.g. [CNS92, Corollaries 7.5 and 7.6].

However if one further looks for $W^{k,p}$ estimates for small enough $k<0$, the canonical solutions will not work. This is due to the ill-posedness of the $\overline{\partial}$-Neumann problem on space of distributions. See [Yao24b, Lemma A.32].

Remark 28 (Near optimal Hölder estimates). If we use (9) for the Hölder spaces, then we have end point optimal Hölder estimates $\mathcal{H}^\Omega:C^{k,\alpha}(\Omega;\wedge^{0,\bullet})\to C^{k,\alpha-}(\Omega;\wedge^{0,\bullet})$ for all $k\ge 0$ and $0<\alpha<1$. This can be done by Sobolev embeddings as follows.

Indeed, for every $\varepsilon>0$ by taking $n/\varepsilon<p<\infty$, we have continuous embeddings $C^{k,\alpha}(\Omega)\hookrightarrow H^{k+\alpha,p}(\Omega)\hookrightarrow C^{k,\alpha-\varepsilon}(\Omega)$, see e.g. [Tri06, Remark 1.96 and Theorem 1.122]. From Remark 8 (ii) we obtain the boundedness $\mathcal{P}^\Omega,\mathcal{H}^\Omega:H^{k+\alpha,p}(\Omega;\wedge^{0,\bullet})\to H^{k+\alpha,p}(\Omega;\wedge^{0,\bullet})$. Thus $C^{k,\alpha}(\Omega;\wedge^{0,\bullet})\to C^{k,\alpha-\varepsilon}(\Omega;\wedge^{0,\bullet})$ is bounded. Letting $\varepsilon\to 0^+$ we get the end point optimal Hölder bounds.

Appendix A. Skew Bergman Projection on Convex Domains of Finite type

In this section we briefly review the construction of homotopy formulae on convex domains of finite type from [Yao24a] and complete the proof to Proposition 26.

Theorem 29. *Let $\Omega\subset\mathbb{C}^n$ be a smooth convex domain of finite type. For the homotopy operators $\mathcal{H}_q:\mathscr{S}'(\Omega;\wedge^{0,q})\to\mathscr{S}'(\Omega;\wedge^{0,q-1})$ for $q=1,\dots,n$ given in* [Yao24a, Theorem 1.1]*, let $\mathcal{P}f:=f-\mathcal{H}_1\overline{\partial}f$ for $f\in\mathscr{S}'(\Omega)$. Then $\mathcal{P}:H^{s,p}(\Omega)\to H^{s,p}(\Omega)$ is bounded for all $s\in\mathbb{R}$ and $1<p<\infty$. In particular $\mathcal{P}:W^{k,p}(\Omega)\to W^{k,p}(\Omega)$ is bounded for $k\in\mathbb{Z}$ and $1<p<\infty$.*

Here for a convex domain we can use affine line type [Yao24a, Definition 3.1] to define the type condition. See e.g. [McN92, BS92] for more discussions.

We briefly review the construction. Let $\varrho:\mathbb{C}^n\to\mathbb{R}$ be a defining function of Ω, which is a smooth function such that $\nabla\varrho(z)\neq 0$ for all $\zeta\in b\Omega$ and $\Omega=\{\zeta\in\mathbb{C}^n:\varrho(\zeta)\neq 0\}$. We can assume that there is a $T_1>0$ such that for all $|t|<T_1$ the sublevel set $\Omega_t:=\{\varrho<t\}$ are all scaled copies of Ω, which in particular have the same finite type as Ω.

Denote $U_1=\{\zeta:-T_1<\varrho(\zeta)<T_1\}$. For each $\zeta\in U_1$, we have orthogonal decomposition of the $(0,1)$ cotangent space $\mathbb{C}^n=T^{*0,1}_\zeta\mathbb{C}^n=(\operatorname{Span}_{\mathbb{C}}\overline{\partial}\varrho(\zeta))\oplus T^{*0,1}_\zeta(b\Omega_{\varrho(\zeta)})$. This leads to an orthogonal decomposition $f=f^\top+f^\perp$ for $(0,q)$ forms $f(\zeta)=\sum_I f_I(z)\overline{\partial}\zeta^I$ defined in U_1:

- $f^\perp$ is in the ideal generated by $\overline{\partial}\varrho$, i.e. $\iota_Z f^\perp=0$ for every $(0,1)$-vector fields $Z=\sum_{j=1}^n Z_j\frac{\partial}{\partial\bar\zeta_j}$ such that $Z\varrho=0$.
- $f^\top$ is a section of $\coprod_\zeta\wedge^q T^{*0,1}_\zeta(b\Omega_{\varrho(\zeta)})$, i.e. $\iota_{\frac{\partial}{\partial\bar\varrho}}f^\top=0$, where $\frac{\partial}{\partial\bar\varrho}=|\overline{\partial}\varrho|^{-2}\sum_{j=1}^n\frac{\partial\varrho}{\partial\zeta_j}\frac{\partial}{\partial\bar\zeta_j}$.

See [Yao24a, Definition 2.6 and Remark 2.8] for details. For a bidegree form $K(z,\zeta)$ in variables z and ζ, we use $K^\top(z,\zeta)$ and $K^\perp(z,\zeta)$ for the projections with respect to ζ-variable but not to z-variable.

For $\zeta\in U_1$ we also define the so-called ε-*minimal ellipsoid* (associated to ϱ):

$$P_\varepsilon(\zeta)=\Big\{\zeta+\sum_{j=1}^n a_jv_j:a_1,\dots,a_n\in\mathbb{C},\ \sum_{j=1}^n\frac{|a_j|^2}{\tau_j(\zeta,\varepsilon)^2}<1\Big\},\tag{12}$$

where $(v_1,\dots,v_n)$ is a unitary basis called ε-*minimal basis* at ζ and $\tau_1(\zeta,\varepsilon)\le\dots\le\tau_n(\zeta,\varepsilon)$ are the side lengths. See [Yao24a, Definition 3.2] and [Hef02, Definition 2.6]. Roughly speaking $\tau_j(\zeta,\varepsilon)$ is the minimum number such that there is a unit vector v_j satisfying $v_j\perp\operatorname{Span}_{\mathbb{C}}(v_1,\dots,v_{j-1})$ and $\varrho(\zeta+\tau_j(\zeta,\varepsilon)\cdot v_j)=\varrho(\zeta)+\varepsilon$. This was first constructed by Yu in [Yu92] that based on the work [Sch91].

Recall from [Yao24a, Lemma 3.3 and Remark 3.4] that the following estimates hold: there is a $C_0>1$ and $\varepsilon_0>0$ such that

(13) For every $0<\varepsilon<\varepsilon_0$ and $P_\varepsilon(\zeta)\subset U_1$, if $z\in P_\varepsilon(\zeta)$ then $\zeta\in P_{C_0\varepsilon}(z)$;
(14) $C_0^{-1}\varepsilon\le\tau_1(\zeta,\varepsilon)\le C_0\varepsilon$ and $\tau_n(\zeta,\varepsilon)\le C_0\varepsilon^{1/m}$, where m is the type of Ω.

See also [Hef02, Section 2] for more details. We shall need the following estimates:

Proposition 30 ([Yao24a, Lemma 3.9])**.** *Let Ω, ϱ, $P_\varepsilon(\zeta)$, $\tau_j(\zeta,\varepsilon)$ and ε_0 be defined as above. There is a neighborhood $\mathcal{U}$ of $\overline{\Omega}$ and a smooth $(1,0)$ form $\widehat{Q}(z,\zeta)=\sum_{j=1}^n Q_j(z,\zeta)d\zeta_j$ defined for $z\in\Omega$ and $\zeta\in\mathcal{U}\backslash\Omega$, such that:*

(i) $\widehat{Q}$ is a Leray form, i.e. $\widehat{Q}$ is holomorphic in z, and $|\widehat{Q}(z,\zeta)|\neq 0$ for all $z\in\Omega$ and $\zeta\in\mathcal{U}\backslash\Omega$.
(ii) Denote $\widehat{S}(z,\zeta):=\sum_{l=1}^n\widehat{Q}_l(z,\zeta)(\zeta_l-z_l)$. For every $k\ge 0$ there is a $C_k>0$ such that for every $0\le j\le n-1$, $0<\varepsilon\le\varepsilon_0$, $\zeta\in\mathcal{U}\backslash\overline{\Omega}$ and $z\in\Omega\cap P_\varepsilon(\zeta)\backslash P_{\varepsilon/2}(\zeta)$,

$$\left|D^k_{z,\zeta}\Big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^j}{\widehat{S}^{j+1}}\Big)^\top(z,\zeta)\right|\le\frac{C_k\varepsilon^{-1-k}}{\prod_{l=2}^{j+1}\tau_l(\zeta,\varepsilon)^2};\quad\left|D^k_{z,\zeta}\Big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^j}{\widehat{S}^{j+1}}\Big)^\perp(z,\zeta)\right|\le\frac{C_k\varepsilon^{-2-k}\tau_{j+1}(\zeta,\varepsilon)}{\prod_{l=2}^{j+1}\tau_l(\zeta,\varepsilon)^2}.\tag{15}$$

Here $D^k=\{\partial_z^\alpha\partial_\zeta^\beta\partial_{\bar\zeta}^\gamma\}_{|\alpha+\beta+\gamma|\le k}$ is the collection of differential operators acting on the components of the forms.

Here in the reference [Yao24a, Lemma 3.9] the second term in (15) is stated for $\big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^j}{\widehat{S}^{j+1}}\big)(z,\zeta)$. Nevertheless using $\big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^j}{\widehat{S}^{j+1}}\big)^\perp=\big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^j}{\widehat{S}^{j+1}}\big)-\big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^j}{\widehat{S}^{j+1}}\big)^\top$ and (14) we get the same estimate (with some larger C_k) for $\big(\frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^j}{\widehat{S}^{j+1}}\big)^\perp(z,\zeta)$.

This Leray map was constructed by Diederich and Fornæss [DF99]. Note that in the original construction [DF99] the support function $S(z,\zeta)$ may have zeros when $|z-\zeta|$ is large. In [Yao24a, Lemma 2.2] we took a standard modification to avoid the issue.

Now we can recall the homotopy operators $(\mathcal{H}_q)_{q=1}^n$ in [Yao24a], which takes the following form:

$$\mathcal{H}_q f(z) = \int_{\mathcal{U}} B_{q-1}(z,\cdot)\wedge \mathcal{E}f + \int_{\mathcal{U}\backslash\overline{\Omega}} K_{q-1}(z,\cdot)\wedge[\overline{\partial},\mathcal{E}]f, \qquad f\in\mathscr{S}'(\Omega;\wedge^{0,q}), \qquad 1\le q\le n. \tag{16}$$

Here $\mathcal{U}$ is the bounded neighborhood of $\overline{\Omega}$ determined in Proposition 30. $\mathcal{E}:\mathscr{S}'(\Omega)\to\mathscr{E}'(\mathcal{U})$ is Rychkov's extension operator [Ryc99], acting on the components of the forms, see [Yao24b, (4.6) and (4.14)] for the precise formula.

$$B(z,\zeta) := \frac{b\wedge(\overline{\partial}b)^{n-1}}{(2\pi i)^n|\zeta-z|^{2n}} = \sum_{q=0}^{n-1} B_q, \quad K(z,\zeta) = \frac{b\wedge\widehat{Q}}{(2\pi i)^n}\wedge\sum_{k=1}^{n-1}\frac{(\overline{\partial}b)^{n-1-k}\wedge(\overline{\partial}\widehat{Q})^{k-1}}{|z-\zeta|^{2(n-k)}(\widehat{Q}\cdot(\zeta-z))^k} = \sum_{q=0}^{n-2}K_q,$$

where $b=\sum_{j=1}^n(\bar\zeta_j-\bar z_j)d\zeta^j$. B is the *Bochner-Martinelli form*, with B_q the component of degree $(0,q)$ in z and $(n,n-q-1)$ in ζ. K is the *Leray-Koppelman form* associated to $\widehat{Q}(z,\zeta)$ in Proposition 30, where K_q is the component of degree $(0,q)$ in z and $(n,n-q-2)$ in ζ.

Denote by

$$F(z,\zeta) := B(z,\zeta) - \overline{\partial}_{z,\zeta}K(z,\zeta), \quad z\in\Omega, \quad \zeta\in\mathcal{U}\backslash\overline{\Omega}, \tag{17}$$

the *Cauchy-Fantappiè form*. Recall from [CS01, Lemma 11.1.1] we have

$$F(z,\zeta) = \frac{\widehat{Q}\wedge(\overline{\partial}\widehat{Q})^{n-1}}{(2\pi i)^n\widehat{S}(z,\zeta)^n} = \frac{\widehat{Q}(z,\zeta)\wedge(\overline{\partial}\widehat{Q}(z,\zeta))^{n-1}}{(2\pi i)^n(\widehat{Q}(z,\zeta)\cdot(\zeta-z))^n}, \quad z\in\Omega, \quad \zeta\in\mathcal{U}\backslash\overline{\Omega}.$$

Note that F is a bi-degree $(n,n-1)$ form, with degree $(0,0)$ in z and $(n,n-1)$ in ζ. We write the decomposition $F=F^\top+F^\perp$ in ζ-variable as defined from above (see [Yao24a, Convention 2.7]).

Proposition 31. *Assume Ω is convex and has finite type m. Let $\delta(w):=\operatorname{dist}(w,b\Omega)$. Then for any $s\in\mathbb{R}$ and $k\in\mathbb{Z}_+$ such that $0<s<k-1$, there is a constant $C=C(\Omega,\mathcal{U},\widehat{Q},k,s)>0$ such that*

$$\int_{\mathcal{U}\backslash\overline{\Omega}}\delta(\zeta)^s|D^k_{z,\zeta}(F^\top)(z,\zeta)|d\operatorname{Vol}(\zeta)\le C\delta(z)^{s+1-k}, \qquad \forall z\in\Omega; \tag{18}$$

$$\int_{\Omega}\delta(z)^s|D^k_{z,\zeta}(F^\top)(z,\zeta)|d\operatorname{Vol}(z)\le C\delta(\zeta)^{s+1-k}, \qquad \forall \zeta\in\mathcal{U}\backslash\overline{\Omega}; \tag{19}$$

$$\int_{\mathcal{U}\backslash\overline{\Omega}}\delta(\zeta)^s|D^k_{z,\zeta}(F^\perp)(z,\zeta)|d\operatorname{Vol}(\zeta)\le C\delta(z)^{s+\frac{1}{m}-k}, \qquad \forall z\in\Omega; \tag{20}$$

$$\int_{\Omega}\delta(z)^s|D^k_{z,\zeta}(F^\perp)(z,\zeta)|d\operatorname{Vol}(z)\le C\delta(\zeta)^{s+\frac{1}{m}-k}, \qquad \forall \zeta\in\mathcal{U}\backslash\overline{\Omega}. \tag{21}$$

As a result if we define for every $\alpha\in\mathbb{N}^{2n}_{\zeta,\bar\zeta}$

$$\mathcal{F}^{\alpha,\top}g(z):=\int_{\mathcal{U}\backslash\overline{\Omega}}D^\alpha_\zeta(F^\top)(z,\cdot)\wedge g, \quad \mathcal{F}^{\alpha,\perp}g(z):=\int_{\mathcal{U}\backslash\overline{\Omega}}D^\alpha_\zeta(F^\perp)(z,\cdot)\wedge g, \qquad g\in L^1(\mathcal{U}\backslash\overline{\Omega};\wedge^{0,1}),$$

then in terms of Definition 7, for every $s>0$ and $1<p<\infty$,

$$\mathcal{F}^{\alpha,\top}:\widetilde{H}^{s,p}(\overline{\mathcal{U}}\backslash\Omega;\wedge^{0,1})\to H^{s+1-|\alpha|,p}(\Omega), \qquad \mathcal{F}^{\alpha,\perp}:\widetilde{H}^{s,p}(\overline{\mathcal{U}}\backslash\Omega;\wedge^{0,1})\to H^{s+\frac{1}{m}-|\alpha|,p}(\Omega) \tag{22}$$

are bounded.

Proof. Notice that for $0<\varepsilon<\varepsilon_0$ and $\zeta\in\mathcal{U}\backslash\Omega$ by (12) we have $|P_\varepsilon(\zeta)\backslash P_{\varepsilon/2}(\zeta)|\le|P_\varepsilon(\zeta)|\le\prod_{l=1}^n\tau_l(\zeta,\varepsilon)^2$. Similarly $|P_\varepsilon(z)\backslash P_{\varepsilon/2}(z)|\le\prod_{l=1}^n\tau_l(z,\varepsilon)^2$ for $-\varepsilon_0<\varrho(z)<0$ as well.

According to Proposition 30, (15) holds for $0<\varepsilon<\varepsilon_0$, $\zeta\in\mathcal{U}\backslash\overline{\Omega}$ and $z\in\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)$. By (13), for a possibly larger $C_k>0$, one also has (15) holds for $0<\varepsilon<\varepsilon_0$, $-\varepsilon_0<\varrho(z)<0$ and $\zeta\in P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cap\Omega)$.

Now take $j=n-1$ in (15), for every $z\in\Omega$ and $\zeta\in\mathcal{U}\backslash\Omega$, we see that

$$\int_{\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)}|D^k(F^\top)(w,\zeta)|d\operatorname{Vol}(w)+\int_{P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)}|D^k(F^\top)(z,w)|d\operatorname{Vol}(w)\le C_k\varepsilon^{1-k}; \tag{23}$$

$$\int_{\Omega\cap P_\varepsilon(\zeta)\backslash P_{\frac{\varepsilon}{2}}(\zeta)}|D^k(F^\perp)(w,\zeta)|d\operatorname{Vol}(w)+\int_{P_\varepsilon(z)\backslash(P_{\frac{\varepsilon}{2}}(z)\cup\Omega)}|D^k(F^\perp)(z,w)|d\operatorname{Vol}(w)\le C_k\varepsilon^{\frac{1}{m}-k}. \tag{24}$$

To prove (18), we note that $0<s<k-1$, and F is bounded and smooth uniformly either for $z\in\Omega$ with $\delta(z)\ge\varepsilon_0$, or for $\zeta\in\mathcal{U}\backslash(P_{\varepsilon_0}(z)\cup\Omega)$. Thus it suffices to show $\int_{P_{\varepsilon_0}(z)\backslash\overline{\Omega}}\delta(\zeta)^s|D^kF^\top(z,\zeta)|d\operatorname{Vol}_\zeta\lesssim\delta(z)^{s+1-k}$, $\forall\delta(z)<\varepsilon_0$. Let $J\in\mathbb{Z}$ be the unique number such that $2^{-J}\varepsilon_0\le\varrho(z)<2^{1-J}\varepsilon_0$. Then $P_{\varepsilon_0}(z)\backslash\overline{\Omega}\subset\cup_{j=1}^JP_{2^{1-j}\varepsilon_0}(z)\backslash(P_{2^{-j}\varepsilon_0}(z)\cup\Omega)$. Applying (23) we get (18):

$$\begin{aligned}\int_{P_{\varepsilon_0}(z)\backslash\overline{\Omega}}\delta(\zeta)^s|D^kF^\top(z,\zeta)|d\operatorname{Vol}_\zeta&\lesssim_k\sum_{j=1}^J\int_{P_{2^{1-j}\varepsilon_0}(z)\backslash(P_{2^{-j}\varepsilon_0}(z)\cup\Omega)}(2^{-j}\varepsilon_0)^s|D^kF^\top(z,\zeta)|d\operatorname{Vol}_\zeta\\&\lesssim_k\sum_{j=1}^J(2^{-j}\varepsilon_0)^s(2^{-j}\varepsilon_0)^{1-k}\lesssim_{\varepsilon_0}2^{-J(s+1-k)}\approx\delta(z)^{s+1-k}.\end{aligned}\tag{25}$$

By swapping z and ζ, the same argument yields (19). Replacing (23) by (24), the same computation as in (25) yields (20) and (21).

The boundedness for $\mathcal{F}^{\alpha,\top}$ is a direct consequence from [Yao24b, Corollary A.28] with (18) and (19), similarly that of $\mathcal{F}^{\alpha,\perp}$ follows from (20) and (21). The proof uses Hardy's distance inequality (see [Yao24a, Proposition 5.3]). □

Proposition 32. *Let $\Omega\subset\mathbb{C}^n$ be a bounded smooth domain and $\mathcal{U}\supset\overline{\Omega}$ be a bounded smooth neighborhood. Let $\mathcal{E}$ be Rychkov's extension operator in* [Yao24a, (4.14)].

(i) For every $k\ge1$ there are linear operators $(\mathcal{S}^{k,\alpha})_{|\alpha|\le k}:\mathscr{S}'(\mathbb{C}^n)\to\mathscr{S}'(\mathbb{C}^n)$ (here $\alpha\in\mathbb{N}^{2n}$) such that $\mathcal{S}^{k,\alpha}:\widetilde{H}^{s,p}(\overline{\mathcal{U}\backslash\Omega})\to\widetilde{H}^{s+k,p}(\overline{\mathcal{U}\backslash\Omega})$ is bounded and $g=\sum_{|\alpha|\le k}D^\alpha\mathcal{S}^{k,\alpha}g$ for all $\operatorname{supp}g\subset\mathcal{U}\backslash\Omega$.

(ii) The map $[f\mapsto([\overline{\partial},\mathcal{E}]f)^\top]:H^{s,p}(\Omega)\to\widetilde{H}^{s-\varepsilon,p}(\overline{\mathcal{U}\backslash\Omega})$ is bounded for all $s\in\mathbb{R}$, $\varepsilon>0$ and $1<p<\infty$.

See [SY24a, Proposition 1.7] for (i) and [Yao24a, Corollary 5.5 (iii)] for (ii).

Proof of Theorem 29. Since $B-F=\overline{\partial}_{z,\zeta}K$ by (17), by separating the degrees we see that $F=B_0-\overline{\partial}_\zeta K_0$. For the same extension operator $\mathcal{E}$ in (16) we have (see e.g. [Gon19, Proposition 2.1]), for $f\in\mathscr{S}'(\Omega)$,

$$\begin{aligned}\mathcal{P}f(z)=&f(z)-\mathcal{H}_1\overline{\partial}f(z)=\mathcal{E}f(z)-\int_{\mathcal{U}}B_0(z,\cdot)\wedge\mathcal{E}\overline{\partial}f-\int_{\mathcal{U}\backslash\overline{\Omega}}K_0(z,\cdot)\wedge[\overline{\partial},\mathcal{E}]\overline{\partial}f\\&=\int_{\mathcal{U}}\overline{\partial}_\zeta B_0(z,\cdot)\wedge\mathcal{E}f-\int_{\mathcal{U}}B_0(z,\cdot)\wedge\overline{\partial}\mathcal{E}f+\int_{\mathcal{U}\backslash\overline{\Omega}}B_0(z,\cdot)\wedge[\overline{\partial},\mathcal{E}]f+\int_{\mathcal{U}\backslash\overline{\Omega}}K_0(z,\cdot)\wedge\overline{\partial}[\overline{\partial},\mathcal{E}]f\\&=\int_{\mathcal{U}\backslash\overline{\Omega}}B_0(z,\cdot)\wedge[\overline{\partial},\mathcal{E}]f-\int_{\mathcal{U}\backslash\overline{\Omega}}\overline{\partial}_\zeta K_0(z,\cdot)\wedge[\overline{\partial},\mathcal{E}]f=\int_{\mathcal{U}\backslash\overline{\Omega}}F(z,\cdot)\wedge[\overline{\partial},\mathcal{E}]f.\end{aligned}$$

Fix $s\in\mathbb{R}$ and $1<p<\infty$. It suffices to show $\mathcal{P}:H^{s,p}(\Omega)\to H^{s,p}(\Omega)$ is bounded.

Take $k \in \mathbb{Z}_+$ such that $k > 1 - s$. By the $\top, \perp$ decomposition, Proposition 32 (i) and integration by parts we have

$$\begin{aligned}\mathcal{P}f(z) &= \int_{\mathcal{U}\backslash\overline{\Omega}} F^\top(z,\cdot) \wedge ([\overline{\partial}, \mathcal{E}]f)^\perp + F^\perp(z,\cdot) \wedge ([\overline{\partial}, \mathcal{E}]f)^\top \\ &= \sum_{|\alpha|\le k} \int_{\mathcal{U}\backslash\overline{\Omega}} F^\top(z,\cdot) \wedge D^\alpha \mathcal{S}^{k,\alpha}\big[([\overline{\partial}, \mathcal{E}]f)^\perp\big] + F^\perp(z,\cdot) \wedge D^\alpha \mathcal{S}^{k,\alpha}\big[([\overline{\partial}, \mathcal{E}]f)^\top\big] \\ &= \sum_{|\alpha|\le k} (-1)^{|\alpha|} \int_{\mathcal{U}\backslash\overline{\Omega}} D_\zeta^\alpha(F^\top)(z,\cdot) \wedge \mathcal{S}^{k,\alpha}\big[([\overline{\partial}, \mathcal{E}]f)^\perp\big] + D_\zeta^\alpha(F^\perp)(z,\cdot) \wedge \mathcal{S}^{k,\alpha}\big[([\overline{\partial}, \mathcal{E}]f)^\top\big] \\ &= \sum_{|\alpha|\le k} (-1)^{|\alpha|} \Big(\mathcal{F}^{\alpha,\top}\mathcal{S}^{k,\alpha}[\overline{\partial}, \mathcal{E}]^\perp + \mathcal{F}^{\alpha,\top}\mathcal{S}^{k,\alpha}[\overline{\partial}, \mathcal{E}]^\top\Big)[f].\end{aligned}$$

Here we use $[\overline{\partial}, \mathcal{E}]^{(\top,\perp)} f := ([\overline{\partial}, \mathcal{E}]f)^{(\top,\perp)}$.

Note that by Proposition 32 (ii) $[\overline{\partial}, \mathcal{E}]^\top : H^{s,p}(\Omega) \to \widetilde{H}^{s-1/m,p}(\overline{\Omega})$ is bounded. On the other hand, since $[\overline{\partial}, \mathcal{E}] : H^{s,p}(\Omega) \to \widetilde{H}^{s-1,p}(\overline{\mathcal{U}\backslash\Omega})$ is clearly bounded and $[\overline{\partial}, \mathcal{E}]^\perp = [\overline{\partial}, \mathcal{E}] - [\overline{\partial}, \mathcal{E}]^\top$, we have the boundedness $[\overline{\partial}, \mathcal{E}]^\perp : H^{s,p}(\Omega) \to \widetilde{H}^{s-1,p}(\overline{\mathcal{U}\backslash\Omega})$. Making use of those, together with the boundedness for $\mathcal{S}^{k,\alpha}$ in Proposition 32 (i), as well as for $\mathcal{F}^{\alpha,\top}$ and $\mathcal{F}^{\alpha,\perp}$ in (22), we apply the following composition arguments: for every $|\alpha| \le k$

$$\begin{aligned}\mathcal{F}^{\alpha,\top}\mathcal{S}^{k,\alpha}[\overline{\partial}, \mathcal{E}]^\perp :& H^{s,p}(\Omega) \xrightarrow{[\overline{\partial},\mathcal{E}]^\perp} \widetilde{H}^{s-1,p}(\overline{\mathcal{U}\backslash\Omega}) \xrightarrow{\mathcal{S}^{k,\alpha}} \widetilde{H}^{s-1+k,p}(\overline{\mathcal{U}\backslash\Omega}) \xrightarrow{\mathcal{F}^{\alpha,\top}} H^{s,p}(\Omega); \\ \mathcal{F}^{\alpha,\perp}\mathcal{S}^{k,\alpha}[\overline{\partial}, \mathcal{E}]^\top :& H^{s,p}(\Omega) \xrightarrow{[\overline{\partial},\mathcal{E}]^\top} \widetilde{H}^{s-1/m,p}(\overline{\mathcal{U}\backslash\Omega}) \xrightarrow{\mathcal{S}^{k,\alpha}} \widetilde{H}^{s-1/m+k,p}(\overline{\mathcal{U}\backslash\Omega}) \xrightarrow{\mathcal{F}^{\alpha,\perp}} H^{s,p}(\Omega).\end{aligned}$$

Taking sums over α we complete the proof. ☐

Acknowledgment. The authors would like to thank Song-Ying Li for some helpful discussion.

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